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Note

Turán–Ramsey problems

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Abstract

For $i = 1,2,\dots,k$, let G_i be a graph with vertex set $[n] = \{1,\dots,n\}$ containing no F_i as a subgraph. At most how many edges are in $G_1 \cup \cdots \cup G_k$? We shall answer this Turán–Ramsey-type question asymptotically, and pose a number of related problems.

Given graphs $F_1,\dots,F_k$, write $\mathrm{ex}_k(n;F_1,\dots,F_k)$ for the maximal size of a graph G of order n whose edges can be coloured with k colours such that G contains no F_i all of whose edges are coloured with the ith colour. Putting it slightly differently, $\mathrm{ex}_k(n;F_1,\dots,F_k)$ is the maximal size of $G_1 \cup \cdots \cup G_k$, where $G_1,\dots,G_k$ are graphs on the same set of n vertices, and no G_i contains F_i as a subgraph. As usual, the graphs F_i are our *forbidden* graphs. A graph G of order n and size $\mathrm{ex}_k(n;F_1,\dots,F_k)$ having an appropriate colouring, is an *extremal graph.*

The problem of determining $\mathrm{ex}_k(n;F_1,\dots,F_k)$ is clearly a Turán–Ramsey-type problem in the sense that it is related to the theorems of Ramsey and Turán. Nevertheless, it is rather different from the Turán–Ramsey problems studied by Erdős and Sós [9], Szemerédi [12], Bollobás and Erdős [3], Erdős, et al. [7], and others.

Our aim in this brief note is to prove some results about $\mathrm{ex}_k(n;F_1,\dots,F_k)$ and pose a number of unsolved problems. We shall follow the standard notation in [1].

In the case $k = 1$, our problem reduces to the quintessential extremal problem: $\mathrm{ex}(n;F) = \mathrm{ex}_1(n;F)$ is the maximal number of edges in a graph of order n that does not contain F as a subgraph. Turán's classical theorem [13] tells us that if F is K_r, the complete graph of order r, then $\mathrm{ex}(n;K_r) = t_{r-1}(n)$ and $T_{r-1}(n)$ is the unique extremal graph. Recall that $T_{r-1}(n)$ is the $(r-1)$-partite *Turán graph* of order n, that is the unique complete $(r-1)$-partite graph of order n with as equal classes as possible, and

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$t_{r-1}(n) = e(T_{r-1}(n))$ is the size of $T_{r-1}(n)$. Trivially, $t_{r-1}(n) \sim (r-2)n^2/2(r-1)$; in fact,

$$\frac{r-2}{r-1}\binom{n}{2} \leqslant t_{r-1}(n) \leqslant \frac{(r-2)n^2}{2(r-1)}.$$

For ease of expression, we shall just talk of *colourings* instead of colourings of the edges, as we shall consider *only* edge-colourings. Also, given $F_1, \dots, F_k$, we shall call a k-colouring of a graph *$(F_1, \dots, F_k)$-permissible*, or simply *permissible*, if no subgraph of our graph which is isomorphic to F_i has all its edges coloured with the ith colour.

As pointed out by Sós, if the forbidden graphs F_i are complete graphs, say $F_i = K_{r_i}$ then $\mathrm{ex}_k(n; F_1, \dots, F_k)$ is easily determined. Indeed, if $k \geqslant 1$ and $r_1, r_2, \dots, r_k \geqslant 2$ then $\mathrm{ex}_k(n; K_{r_1}, \dots, K_{r_k}) = t_{r-1}(n)$, where r is the Ramsey number $R_k(r_1, \dots, r_k)$.

To see this, let G be a graph of order n and size $t_{r-1}(n)$ that has a permissible k-colouring. The graph G does not contain K_r as a subgraph since K_r itself fails to have a permissible k-colouring. But then, by Turán's theorem, G is (isomorphic to) the Turán graph $T_{r-1}(n)$. This shows that $\mathrm{ex}_k(n; K_{r_1}, \dots, K_{r_k}) \leqslant t_{r-1}(n)$ and $T_{r-1}(n)$ is the only possible extremal graph. But this is an immediate consequence of the fact that, by the definition of a Ramsey number, K_{r-1} has a permissible k-colouring. Indeed, let K_{r-1} have vertex set $\{v_1, \dots, v_{r-1}\}$, let $V_1, \dots, V_{r-1}$ be the vertex classes of a $T_{r-1}(n)$. Take a permissible k-colouring of K_{r-1}, and define a k-colouring of $T_{r-1}(n)$ by colouring all $V_i - V_j$ edges with the colour of the edge $v_i v_j$. Trivially, this is a permissible k-colouring of $T_{r-1}(n)$.

As usual, write $R_k(F_1, \dots, F_k)$ for the *Ramsey number* of the sequence $F_1, \dots, F_k$, that is for the minimal integer r for whch K_r has no permissible k-colouring. Equivalently, $R_k(F_1, \dots, F_k) = \min\{n : \mathrm{ex}_k(n; F_1, \dots, F_k) < \binom{n}{2}\}$.

The first part of the reasoning above implies that

$$\mathrm{ex}_k(n; F_1, \dots, F_k) \leqslant t_{r-1}(n),$$

where $r = R_k(F_1, \dots, F_k)$, no matter what the graphs $F_1, \dots, F_k$ are. But are we close to equality here? It does not take very long to realize that we are not. For example, for $k = 2$ and $F_1 = F_2 = K_3(10)$, the Ramsey number $R_2(F_1, F_2)$ is rather large, certainly more than 3000 but, as we shall see, $\mathrm{ex}_2(n; F_1, F_2)$ is about $t_5(n)$, which is much smaller than $t_{3000}(n)$. As in the case of $\mathrm{ex}(n; F)$, the order of $\mathrm{ex}_k(n; F_1, \dots, F_k)$ depends only on the chromatic numbers $\chi(F_1), \dots, \chi(F_k)$ of the graphs $F_1, \dots, F_k$. To deduce this from the classical Erdős–Stone theorem [10], we shall make use of the next theorem, which is of interest in its own right.

Let H be an n by n bipartite graph that contains no $K(s,t)$, with s vertices in the first class and t in the second. If every vertex in the second class has degree at least d then, rather trivially,

$$(t-1)\binom{n}{s} \geqslant m\binom{d}{s}. \tag{1}$$

[See inequality (2) on p. 73 of [1].]

Before stating Theorem 1, recall that $K_r(n)$ is a complete r-partite graph, with n vertices in each of the r classes. (In particular, $K_r(n) = T_r(rn)$.) Also, given an edge-coloured r-partite graph H with vertex classes $V_1, \ldots, V_r$, call H *weakly monochromatic* if, for each pair of classes (V_i, V_j), all $V_i - V_j$ edges have the same colour.

Theorem 1. *Given $r \geqslant 2$ and $k \geqslant 2$, if n is sufficiently large then every k-colouring of the edges of $K_r(n)$ contains a weakly monochromatic complete r-partite graph, with at least $n^{1/r}$ vertices in the first class, and at least $\log n/r^2 \log k$ vertices in each of the other classes.*

Proof. Set $s = \lceil \log n/r^2 \log k \rceil$ and $t = \lceil n^{(r-1)/r} \rceil$. Let $V_1, V_2, \ldots, V_r$ be the vertex classes of our $K_r(n)$, with $V_i = \{x_1^{(i)}, x_2^{(i)}, \ldots, x_n^{(i)}\}, i = 1, 2, \ldots, n$. For each $j, 1 \leqslant j \leqslant n$, and each vertex $x_h^{(r)}$, the $k-1$ edges $x_h^{(r)} x_j^{(1)}, x_h^{(r)} x_j^{(2)}, \ldots, x_h^{(r)} x_j^{(r-1)}$ are coloured in one of k^{r-1} ways. Therefore, for each $j, 1 \leqslant j \leqslant n$, there is a set $W_j \subset V_r$ with $|W_j| = m = \lceil n/k^{r-1} \rceil$ such that all the edges from $x_j^{(i)}$ to W_j have the same colour $c_j^{(i)}$. Furthermore, as there are k^{r-1} choices for $(c_j^{(1)}, c_j^{(2)}, \ldots, c_j^{(r-1)})$, for some set of at least m values of j these $(r-1)$-tuples are the same. We may assume that $c_j^{(i)} = c^{(i)}$ for $j = 1, 2, \ldots, m$.

Let us define an n by m bipartite graph H_r with bipartition (V_r, U_r), where $U_r = \{u_1, u_2, \ldots, u_m\}$, and in which u_j is joined to the vertices in W_j. We claim that

$$(t-1)\binom{n}{s} < m\binom{m}{s}, \tag{2}$$

and so, by inequality (1), H_r contains a $K(s,t)$. To check (2), it suffices to check that

$$\frac{t-1}{m} < \left(\frac{m-s}{n}\right)^s. \tag{3}$$

But if n is sufficiently large then

$$\left(\frac{m-s}{n}\right)^s = \left(1 - \frac{s}{m}\right)^s \left(\frac{m}{n}\right)^s \geqslant \frac{1}{2}(1/k^{r-1})^{\log n/r^2 \log k + 1}$$
$$\geqslant (2k^{r-1})^{-1} n^{-(r-1)/r^2} > k^{r-1} n^{-1/r} > \frac{t-1}{m},$$

so (3) and (2) do hold. Thus, H_r contains a $K(s,t)$, as claimed.

This implies that our original coloured $K_r(n)$ contains sets $T_1 \subset V_1, \ldots, T_{r-1} \subset V_{r-1}$ and $S_r \subset V_r$ such that all $T_i - S_r$ edges have the same colour $c^{(i)}, |T_1| = \cdots = |T_{r-1}| = t$ and $|S_r| = s$.

For $r = 2$ this proves the assertion. Suppose then that $r \geqslant 3$ and the theorem holds for smaller values of r. The subgraph of our $K_r(n)$ spanned by $T_1 \cup \cdots \cup T_{r-1}$ is a k-coloured $K_{r-1}(t)$, so it contains a weakly monochromatic complete graph with vertex set $S_1 \cup \cdots \cup S_{r-1}$, with $|S_1| \geqslant t^{1/(r-1)}$ and $|S_i| \geqslant \log t/(r-1)^2 \log k$ for $2 \leqslant i \leqslant r-1$. But then the complete r-partite subgraph of $K_r(n)$ spanned by $S_1 \cup \cdots \cup S_r$ has the required properties. Indeed, we have constructed it to be weakly monochromatic, and

$$|S_1| \geqslant t^{1/(r-1)} \geqslant n^{1/r},$$

$$|S_i| \geqslant \log t/(r-1)^2 \log k \geqslant \log n/r(r-1) \log k$$

for $2 \leqslant i \leqslant r-1$, and

$$|S_r| = s \geqslant \log n/r^2 \log k,$$

as required. □

The bound $n^{1/r}$ in the result above is rather arbitrary: given $\varepsilon > 0$, it can be replaced by $n^{1-\varepsilon}$ at the expense of replacing $1/(r^2 \log k)$ by a smaller constant.

Extending the classical Erdős–Stone theorem, Bollobás and Erdős [2] proved that if $r \geqslant 2, \varepsilon > 0$ and n is sufficiently large then every graph of order n and size at least$((r-2)/(r-1)) + \varepsilon)\binom{n}{2}$ contains at $K_r(t)$ with $t \geqslant c \log n$, where $c = c(r, \varepsilon) > 0$ depends only on r and ε. (For further extensions, see [4, 5].) Combining this with Theorem 1, we find the following result.

Theorem 2. *Let $r \geqslant 2$ and $\varepsilon > 0$ be fixed. If n is sufficiently large then every k-coloured graph G of order n and size at least $((r-2)/(r-1)) + \varepsilon)\binom{n}{2}$ contains a weakly monochromatic $K_r(l)$, with $l \geqslant \log \log n/(r+1)^2 \log k$.*

Proof. As remarked above, G contains a $K_r(t)$ with $t \geqslant c(r, \varepsilon) \log n$. By Theorem 1, every k-colouring of this $K_r(t)$ contains a weakly monochromatic $K_r(l)$ with $l \geqslant \log t/r^2 \log k$. If n is sufficiently large, $l \geqslant \log \log n/(r+1)^2 \log k$, as claimed. □

Analogous to the observation of Erdős and Simonovits [8] that the Erdős–Stone theorem implies immediately the asymptotic size of $\mathrm{ex}(n; F)$, the theorem above tells us roughly how large $\mathrm{ex}_k(n; F_1, \ldots, F_k)$ is: if $\chi(F_i) = r_i, i = 1, \ldots, k$, and $r = R_k(r_1, \ldots, r_k)$ then

$$\lim_{n \to \infty} \mathrm{ex}_k(n; F_1, \ldots, F_k) \Big/ \binom{n}{2} = \frac{r-2}{r-1}. \tag{4}$$

In fact, the deduction of (4) from Theorem 2 is somewhat of an overkill since all one needs is that every k-coloured graph G of order n and size at least

$$((r-2)/(r-1)) + \varepsilon)\binom{n}{2}$$

contains a weakly monochromatic $K_r(l)$, with $l \to \infty$ as $n \to \infty$ (see [6]).

As always, relation (4) is easily extended to the case when F_i is replaced by a family $\mathscr{F}_i$ of forbidden subgraphs, and r_i is defined to be $\min_{F \in \mathscr{F}_i} \chi(F)$.

It would be good to sharpen Theorems 1 and 2. Keeping r and k constant, Theorem 1 is not far from being best possible: writing $s(n; r, k)$ for the maximal value of s such that every k-colouring of $K_r(n)$ contains a weakly monochromatic $K_r(s)$, Theorem 1 claims that $s(n;r,k) \geqslant \log n/r^2 \log k$, and it is easily seen that $s(n;r,k) \leqslant 2\log n/(r-1)\log k$. It is likely that, in fact, this upper bound gives the correct order of $s(n;r,k)$.

More importantly, for $r \geqslant 2, k \geqslant 2, 0 < \varepsilon < 1/(r-1)$ and n large, write $l(n;r,k,\varepsilon)$ for the maximal value of l such that every k-coloured graph of order n and size at least $((r-2)/(r-1))+\varepsilon)\binom{n}{2}$ contains a weakly monochromatic $K_r(l)$. Theorem 2 claims that $l(n;r,k,\varepsilon) \geqslant \log\log n/(r+1)^2 \log k$, but this seems to be far from being best possible. An easy argument shows that $l(n;r,k,\varepsilon) \leqslant c(r,k,\varepsilon)\log n$, more or less as in the upper bound in the Erdős–Stone theorem (see [8]). It is likely that the correct order is, once again, given by the upper bound. However, as this would be a substantial extension of the *quantitative* form of the Erdős–Stone theorem, as given in [2, 3, 5], it is probably rather difficult to prove.

Finally, let us pose a common generalization of the problems discussed above and of the original Turán–Ramsey-type problems, as discussed in [9, 12, 3, 7]. Let H be a graph of order n, and let $F_1,\ldots,F_k$ and $F_1',\ldots,F_l'$ be graphs of order at most n. Let $\mathrm{ex}_{k,l}(H;F_1,\ldots,F_k;F_1',\ldots,F_l')$ be the maximal size of a subgraph G of H such that G has an $(F_1,\ldots,F_k)$-permissible k-colouring and $H-E(G)$, the complement of G in H, has an $(F_1',\ldots,F_l')$-permissible l-colouring. Determine or at least estimate $\mathrm{ex}_{k,l}(H;F_1,\ldots,F_k;F_1',\ldots,F_l')$ for a wide class of parameters. Needless to say, this function is defined only if H has an $(F_1,\ldots,F_k,F_1',\ldots,F_l')$-permissible $(k+l)$-colouring.

Setting $H=K_n, k=1, F_1=K_r, l=1$, and $F_1'=K_s$, where $s=o(n)$, we regain the original Turán–Ramsey problems, and for $H=K_n, l=0$ (or $l=0$ and $F_1'=K_n$) we regain the problems discussed above. The general problem is clearly intractable, but there is no doubt that many special cases will amply repay their study.

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